

On the probability that two elements of a finite semigroup¹ have the same right matrix

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Abstract

Let σ be a binary relation on a non empty finite set A . Let $P_\sigma(A)$ denote the probability that a randomly selected couple $(a, b) \in A \times A$ belongs to σ . In this paper we investigate $P_\sigma(A)$ in special cases.

1 Introduction and motivation

Let A be a finite set with a binary operation. Let $Pr(A)$ denote the probability that a pair of elements of A commute with each other. In [3], it is proved that, for any non commutative finite group A , $Pr(A) \leq \frac{5}{8}$. A similar result is proved for rings. In [5], it is proved that, for any non commutative finite ring A , $Pr(A) \leq \frac{5}{8}$. In paper [1], $Pr(A)$ was investigated for finite semigroups A . Contrary to the above mentioned results, it was proved that $Pr(A) > \frac{5}{8}$ for some non commutative finite semigroup A . In fact, there are non commutative finite semigroups A , for which the probability $Pr(A)$ is arbitrarily closed to 1.

Let A be a non empty finite set and σ be a binary relation on A . Generalizing the above investigations, it is a natural idea to answer the following question. What is the probability that a randomly selected couple $(a, b) \in A \times A$ belongs to σ ? In Section 2, we prove a result on the probability $P_\sigma(A)$ when A is a finite set and σ is an equivalence relation A . In Section 3, we deal with $P_\sigma(A)$ in that case when A is a finite semigroup and σ is the kernel of the right regular representation of A . In Section 4, we extend some results of Section 3 for that case when σ is an arbitrary congruence on a finite semigroup A .

2 Equivalence relations

Theorem 2.1 *If σ is an equivalence relation on a non empty finite set A then $P_\sigma(A) \geq \frac{1}{|A/\sigma|}$. The equation $P_\sigma(A) = \frac{1}{|A/\sigma|}$ holds if and only if the cardinality of any two σ -classes of A is the same.*

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Proof. Let m denote the cardinality of the factor set A/σ . Let A_i ($i = 1, \dots, m$) denote the pairwise different σ -classes of A . If $|A_i| = t_i$ then

$$P_\sigma(A) = \frac{t_1^2 + \dots + t_m^2}{(t_1 + \dots + t_m)^2}.$$

By the well known connection between the root mean square and the arithmetic mean, we have

$$\sqrt{\frac{\sum_{i=1}^m t_i^2}{m}} \geq \frac{\sum_{i=1}^m t_i}{m},$$

that is

$$\frac{\sum_{i=1}^m t_i^2}{m} \geq \frac{(\sum_{i=1}^m t_i)^2}{m^2}$$

from which we get

$$P_\sigma(A) = \frac{t_1^2 + \dots + t_m^2}{(t_1 + \dots + t_m)^2} \geq \frac{1}{m} = \frac{1}{|A/\sigma|}.$$

The equation $P_\sigma(A) = \frac{1}{|A/\sigma|}$ holds if and only if

$$\frac{t_1^2 + \dots + t_m^2}{(t_1 + \dots + t_m)^2} = \frac{1}{m},$$

that is

$$\sqrt{\frac{\sum_{i=1}^m t_i^2}{m}} = \frac{\sum_{i=1}^m t_i}{m}$$

which is satisfied if and only if

$$t_1 = \dots = t_m.$$

□

3 Right regular representation of finite semi-groups

A non empty set with an associative binary operation is called a semigroup. Let A be a semigroup and a be an arbitrary element of A . Let ϱ_a denote the inner right translation of A defined by a . Recall that ϱ_a is defined by $(x)\varrho_a = xa$ for every $x \in A$. It is known that $a \mapsto \varrho_a$ is a homomorphism of A into the semigroup of all right translation of A ([2]). This homomorphism is called the right regular representation of the semigroup A .

Let A and B be arbitrary non-empty sets. By an A -matrix over B we mean a mapping of the Cartesian product $A \times A$ into B .

Let $G^0 = \{1, 0\}$ be a group $G = \{1\}$ with a zero 0 adjoined. For an element a of a semigroup A , let $R^{(a)}$ denote the A -matrix over G^0 defined by

$$R^{(a)}((x, y)) = \begin{cases} 1, & \text{if } xa = y \\ 0 & \text{otherwise.} \end{cases}$$

This matrix is called the right matrix over G^0 defined by a . (The right matrices were investigated, for example, in [8] and [9].) It is known that $a \mapsto R^{(a)}$ is a homomorphism of a semigroup A into the multiplicative semigroup of all strictly row-monomial A -matrices over G^0 (see, for example, Exercise 4(a) for §3.5 of [2]). It is easy to see that, for every elements a and b of A , $R^{(a)} = R^{(b)}$ if and only if $\varrho_a = \varrho_b$.

Let θ denote the binary relation on a semigroup A defined by

$$\theta = \{(a, b) \in A \times A : (\forall x \in A) \, xa = xb\}.$$

It is clear that $(a, b) \in \theta$ if and only if the right matrices over G^0 defined by a and b are the same.

In this section, we investigate the probability that two randomly selected elements of a finite semigroup A have the same right matrix, that is, we investigate the probability

$$P_\theta(A) = \frac{|\{(a, b) \in A \times A : (a, b) \in \theta\}|}{|A \times A|}.$$

A semigroup A is said to be left reductive if, for every $a, b \in A$, the assumption " $xa = xb$ for every $x \in A$ " implies $a = b$ ([6]). It is clear that a semigroup A is left reductive if and only if $\theta = \iota_A$, where ι_A denotes the identity relation on A . Thus $P_\theta(A) = \frac{1}{|A|}$ for every left reductive finite semigroups A . In the next, we prove that the converse assertion is not true, in general. We prove that there is a non left reductive finite semigroup A with $P_\theta(A) = \frac{1}{|A|}$.

Assertion (a) of Theorem 2 of [10] shows how we can construct non left reductive semigroups by the help of left reductive ones. We cite this result.

Lemma 3.1 *Let T be a semigroup. For each $x \in T$, associate a set $A_x \neq \emptyset$. Assume that $A_x \cap A_y = \emptyset$ for all $x \neq y$. Assume that, for each triple $(x, y, xy) \in T \times T \times T$ is associated a mapping $f_{(x, y, xy)}$ of A_x into A_{xy} acting on the right. Suppose further that, for every $(x, y, z) \in T \times T \times T$ and every $a \in A_x$,*

$$(a)(f_{(x, y, xy)} \circ f_{(xy, z, xyz)}) = (a)f_{(x, yz, xyz)}.$$

Given $a, b \in \cup_{x \in T} A_x = A$, set

$$a \star b = (a)f_{(x, y, xy)},$$

where $a \in A_x$ and $b \in A_y$. Then $(A; \star)$ is a semigroup, and each set A_x is contained in a θ -class of A . If T is left reductive then the θ -classes of A are exactly the sets A_x ($x \in T$). \square

If, in Lemma 3.1, T is a left reductive semigroup and at least one of the sets A_x ($x \in T$) contains more than one element then the semigroup $(A; \star)$ is not left reductive.

In our investigation, the following construction will play an important role. This construction is a special case of the construction in Lemma 3.1.

A semigroup A is called a left cancellative semigroup if $xa = xb$ implies $a = b$ for every $a, b, x \in A$. It is clear that every left cancellative semigroup is left reductive.

Construction 3.1 (Construction 1 of [11]) Let T be a left cancellative semigroup. For each $x \in T$, associate a nonempty set A_x such that $A_x \cap A_y = \emptyset$ for every $x \neq y$. As T is left cancellative, $x \mapsto tx$ is an injective mapping of T onto tT .

For arbitrary couple $(t, r) \in T \times T$ with $r \in tT$, let $f_{(t,r)}$ be a mapping of A_t into A_r . For all $t \in T$, $r \in tT$, $q \in rT \subseteq tT$ and $a \in A_t$, assume

$$(a)(f_{(t,r)} \circ f_{(r,q)}) = (a)f_{(t,q)}.$$

On the set $A = \cup_{t \in T} A_t$ define an operation \star as follows: for arbitrary $a \in A_t$ and $b \in A_x$, let

$$a \star b = (a)f_{(t,tx)}.$$

By Lemma 3.1, $(A; \star)$ is a semigroup, and each set A_x ($x \in T$) is a θ -class of A . \square

In the next we consider a special type of semigroups (Construction 3.2) which can be obtained by using Construction 3.1.

A semigroup A is called a right zero semigroup if $ab = b$ is satisfied for every $a, b \in A$ ([6]). It is clear that a right zero semigroup is left cancellative.

Construction 3.2 Let m be a positive integer. Let T be a right zero semigroup with $|T| = m$. Let A_x ($x \in T$) be non empty sets such that $A_x \cap A_y = \emptyset$ for all $x, y \in T$ with $x \neq y$. Fix an element x^* in A_x for every $x \in T$. For every $x, y \in T$, let $f_{x,y}$ be the mapping of A_x into $A_{xy} = A_y$ defined by $(a)f_{x,y} = y^*$. It is easy to see that this system of mappings satisfies the conditions of Construction 3.1. Let $A^{(m)}$ be the semigroup which can be obtained from the right zero semigroup T , the sets A_x ($x \in T$) and the mappings $f_{(x,y)}$ ($x, y \in T$) by using Construction 3.1. \square

Theorem 3.1 For any positive integer m , there is a non left reductive finite semigroup A such that $P_\theta(A) = \frac{1}{m}$.

Proof. We shall use Construction 3.2. Let m be an arbitrary positive integer. Let T be a right zero semigroup with $|T| = m$. Let A_x ($x \in T$) be pairwise bijective finite sets containing $k \geq 2$ elements such that $A_x \cap A_y = \emptyset$ for all $x, y \in T$ with $x \neq y$. Fix an element x^* in A_x for every $x \in T$. For every

$x, y \in T$, let $f_{x,y}$ be the mapping of A_x into A_y defined as in Construction 3.2. Let $A^{(m,k)}$ denote the semigroup which can be obtained from the right zero semigroup T , the sets A_x ($x \in T$) and the mappings $f_{(x,y)}$ ($x, y \in T$) by using Construction 3.2. As the θ -classes A_x ($x \in T$) of $A^{(m,k)}$ has the same cardinality $k \geq 2$, $A^{(m,k)}$ is not left reductive, and $P_\theta(A^{(m,k)}) = \frac{1}{m}$ by Theorem 2.1 and Construction 3.1 (regardless of the choice of k). \square

For every positive integer m , consider a semigroup $A^{(m,k)}$ defined in the proof of Theorem 3.1. Then (regardless of the choice of k)

$$\lim_{m \rightarrow \infty} P_\theta(A^{(m,k)}) = 0$$

and so there are non left reductive finite semigroups A , for which the probability $P_\theta(A)$ is arbitrarily closed to 0.

In the next, we show that there are non left reductive finite semigroups A , for which the probability $P_\theta(A)$ is arbitrarily closed to 1.

Theorem 3.2 *For every positive integer k , there is a non left reductive finite semigroups $A^{(k)}$ such that*

$$\lim_{k \rightarrow \infty} P_\theta(A^{(k)}) = 1.$$

Proof. Let m be a fixed positive integer with $m \geq 2$. For every positive integer k , let $A^{(k)}$ denote a semigroup which is defined as in Construction 3.2 under that condition that (T is an m -element right zero semigroup, and) the sets A_x ($x \in T$) satisfy the following: $|A_{x_0}| = k + 1$ for an element $x_0 \in T$, and $|A_x| = 1$ for every $x \in T \setminus \{x_0\}$. As the θ -classes of the semigroup $A^{(k)}$ are the sets A_x ($x \in T$), it is clear that

$$P_\theta(A^{(k)}) = \frac{(k+1)^2 + m - 1}{(m+k)^2}$$

for every k . It is a matter of checking to see that

$$\lim_{k \rightarrow \infty} \frac{(k+1)^2 + m - 1}{(m+k)^2} = 1$$

which proves

$$\lim_{k \rightarrow \infty} P_\theta(A^{(k)}) = 1.$$

\square

In the next we characterize finite semigroups A for which $P_\theta(A) = 1$.

A semigroup A is called a left zero semigroup if $ab = a$ is satisfied for every $a, b \in A$ ([6]). We note that $\theta = \omega_A$ for every left zero semigroup A , where ω_A denotes the universal relation on A . By a zero semigroup we mean a semigroup A with a zero 0 in which $ab = 0$ for every $a, b \in A$.

Construction 3.3 Let L be a left zero semigroup and Q be a zero semigroup. Let Q^* denote Q if $|Q| = 1$. In case $|Q| > 1$, let Q^* denote the partial semigroup $Q \setminus \{0\}$, where 0 denotes the zero of Q . Assume $L \cap Q^* = \emptyset$. Let φ be a mapping (acting on the right) of the set $A = L \cup Q^*$ into L which leaves the elements of L fixed. Let \diamond denote the following operation on A : for arbitrary $a, b \in A$, let

$$a \diamond b = (a)\varphi.$$

Then, for every $a, b, c \in A$, we have

$$(a \diamond b) \diamond c = ((a)\varphi) \diamond c = ((a)\varphi)\varphi = (a)\varphi$$

and

$$a \diamond (b \diamond c) = a \diamond (b)\varphi = (a)\varphi.$$

Consequently $(A; \diamond)$ is a semigroup. \square

Let I be an ideal of a semigroup A . Then

$$\varrho_I = \{(a, b) \in A \times A : a = b \text{ or } a, b \in I\}$$

is a congruence on A . This congruence is called the Rees congruence on A modulo I . The equivalence classes of ϱ_I are I itself and every one-element set $\{a\}$ with $a \in A \setminus I$. The factor semigroup of A modulo ϱ_I is called the Rees factor semigroup of A modulo I which is denoted by A/I . We may describe A/I as the result of collapsing I into the zero element of A/I , while the elements of A outside of I retain their identity.

We say that a semigroup A is an ideal extension of a semigroup L by a semigroup Q with a zero if A has an ideal I which is isomorphic to L and the factor semigroup A/I is isomorphic to Q . We note that the semigroup $(A; \circ)$ defined in Construction 3.3 is an ideal extension of the semigroup L by the semigroup Q .

Theorem 3.3 *For a finite semigroup A , $P_\theta(A) = 1$ is satisfied if and only if A is isomorphic to a semigroup defined in Construction 3.3.*

Proof. Assume $P_\theta(A) = 1$. Then $xa = xb$ for every $x, a, b \in A$. Consequently, for every $a \in A$, we have

$$a^2 = aa = aa^2 = a^3$$

and so a^2 is an idempotent element. Let $E(A)$ denote the set of all idempotent elements of A . If $e, f \in E(A)$ then

$$ef = ee = e$$

and so $E(A)$ is a left zero semigroup. Let $a, b \in A$ be arbitrary elements. Then

$$ab = aa^2 = a^3 = a^2 \in E(A)$$

from which it follows that $E(A)$ is an ideal of A and the Rees factor semigroup $A/E(A)$ is a null semigroup. Thus A is an ideal extension of the left zero semigroup $E(A)$ by the null semigroup $A/E(A)$. We can suppose that $E(A) \cap (A/E(A))^* = \emptyset$. It is clear that $A = E(A) \cup (A/E(A))^*$. Let $a \in A$ be an arbitrary element. Then $|aA| = 1$. Let $(a)\varphi$ denote the element of aA . Then $\varphi : A \mapsto L$ defined by

$$\varphi : a \mapsto (a)\varphi$$

is a mapping of A onto $E(A)$ which leaves the elements of $E(A)$ fixed. As

$$a \diamond b = (a)\varphi = ab$$

for every $a, b \in A$, the semigroup A is isomorphic to the semigroup $(A; \diamond)$ which can be obtained from $L = E(A)$ and $Q = A/E(A)$ by using Construction 3.3.

Conversely, in the semigroup defined in Construction 3.3,

$$a \diamond b = (a)\varphi = a \diamond c$$

for every $a, b, c \in A$ and so $P_\theta(A) = 1$. □

For an arbitrary congruence σ of a semigroup A , we consider the sequence

$$\sigma^{(0)} \subseteq \sigma^{(1)} \subseteq \dots \subseteq \sigma^{(n)} \subseteq \dots$$

of congruences on A , where $\sigma^{(0)} = \sigma$ and, for an arbitrary non-negative integer n , $\sigma^{(n+1)}$ is defined by $(a, b) \in \sigma^{(n+1)}$ if and only if $(xa, xb) \in \sigma^{(n)}$ for every $x \in A$.

The congruences $\sigma^{(n)}$ will be investigated in Section 4 for arbitrary congruences σ on finite semigroups. Here we deal with that case when $\sigma = \theta$. We note that $\theta = \iota_A^{(1)}$ and so $\theta^{(n)} = \iota_A^{(n+1)}$ for every non negative integer n .

A semigroup N with a zero 0 is called a nilpotent semigroup if $N^k = \{0\}$ for some positive integer k ([6]).

Theorem 3.4 *For a finite semigroup A , $P_{\theta^{(n)}}(A) = 1$ for some non-negative integer n if and only if A is an ideal extension of a left zero semigroup by a nilpotent semigroup.*

Proof. Let A be a finite semigroup such that $P_{\theta^{(n)}}(A) = 1$ for some non-negative integer n . Then $P_{\iota_A^{(n+1)}}(A) = 1$, that is, $\iota_A^{n+1} = \omega_A$. By Theorem 5 of [7], A is an ideal extension of a left zero semigroup by a nilpotent semigroup.

Conversely, assume that a finite semigroup A is an ideal extension of a left zero semigroup by a nilpotent semigroup. Then, by Theorem 5 of [7], $\iota_A^{(n)} = \omega_A$ for some non-negative integer n . If $n = 0$ then $\iota_A = \omega_A$ and so $|S| = 1$; in this case $P_{\theta^{(n)}}(A) = 1$ for every non-negative integer n . If $n \geq 1$ then $\omega_A = \iota_A^{(n)} = \theta^{(n-1)}$ and so, by Theorem 5 of [7], $P_{\theta^{(n-1)}}(A) = 1$. □

4 Arbitrary congruences on finite semigroups

Let α and β be arbitrary congruences on a semigroup A with

$$\alpha \subseteq \beta.$$

Let β/α be the relation on the factor semigroup A/α defined by: for some α -classes $[a]_\alpha, [b]_\alpha \in S/\alpha$,

$$([a]_\alpha, [b]_\alpha) \in \beta/\alpha$$

if and only if

$$(a, b) \in \beta.$$

By Theorem 5.6 of [4] and I.3.3 Lemma of [12], β/α is a congruence on A/α , and

$$\Phi_\alpha : \beta \mapsto \beta/\alpha$$

is a lattice isomorphism of the lattice of all congruences on A containing α onto the lattice of all congruences of the factor semigroup S/α . It is clear that

$$\Phi_\alpha(\alpha) = \iota_{A/\alpha}, \quad \Phi_\alpha(\omega_A) = \omega_{A/\alpha}.$$

Let σ be an arbitrary congruence on a semigroup A . For every $a, b \in A$,

$$([a]_\sigma, [b]_\sigma) \in \sigma^{(n)}/\sigma = \Phi_\sigma(\sigma^{(n)}) \quad (n \geq 1)$$

if and only if

$$(xa, xb) \in \sigma \quad \text{for every } x \in A^n,$$

that is,

$$[x]_\sigma[a]_\sigma = [x]_\sigma[b]_\sigma \quad \text{for every } x \in A^n.$$

Denoting the kernel of the right regular representation on the factor semigroup A/σ by $\theta_{A/\sigma}$, the last condition is equivalent to the condition that

$$([x']_\sigma[a]_\sigma, [x']_\sigma[b]_\sigma) \in \theta_{A/\sigma} \quad \text{for every } [x']_\sigma \in (A/\sigma)^{(n-1)},$$

that is,

$$([a]_\sigma, [b]_\sigma) \in \theta_{A/\sigma}^{(n-1)}.$$

Consequently,

$$\Phi_\sigma(\sigma^{(n)}) = \theta_{A/\sigma}^{(n-1)}$$

for every positive integer n . Especially,

$$\Phi_\sigma(\sigma^{(1)}) = \theta_{A/\sigma}.$$

Theorem 4.1 *For a finite semigroup A and an arbitrary congruence σ on A , $P_{\sigma^{(1)}}(A) = 1$ is satisfied if and only if the factor semigroup A/σ is isomorphic to a semigroup defined in Construction 3.3.*

Proof. For a finite semigroup A and an arbitrary congruence σ on A , $P_{\sigma^{(1)}}(A) = 1$ is satisfied if and only if $\sigma^{(1)} = \omega_A$, that is,

$$\theta_{A/\sigma} = \Phi_{\sigma}(\sigma^{(1)}) = \Phi_{\sigma}(\omega_A) = \omega_{A/\sigma}$$

which is equivalent to the condition that $P_{\theta_{A/\sigma}}(A/\sigma) = 1$. Using Theorem 3.3, this last equation is equivalent to the condition that the factor semigroup A/σ is isomorphic to a semigroup defined in Construction 3.3. \square

Theorem 4.2 *For a finite semigroup A and an arbitrary congruence σ on A , $P_{\sigma^{(n)}}(A) = 1$ for some positive integer n if and only if the factor semigroup A/σ is an ideal extension of a left zero semigroup by a nilpotent semigroup.*

Proof. As $P_{\sigma^{(n)}}(A) = 1$ if and only if $\sigma^{(n)} = \omega_A$, the condition that $P_{\sigma^{(n)}}(A) = 1$ is satisfied for some positive integer n is equivalent to the condition that

$$\theta_{A/\sigma}^{(m)} = \omega_{A/\sigma}$$

is satisfied for some non negative integer m . By Theorem 3.4, it is equivalent to the condition that the factor semigroup A/σ is an ideal extension of a left zero semigroup by a nilpotent semigroup. \square

References

- [1] AHMEDIDELIR, K., CAMPBELL, C.M, DOOSTIE, H. (2011) Almost commutative semigroups. *Algebra Colloquium* **18** 881–888.
- [2] CLIFFORD, A.H. and PRESTON, G.B. (1961) *The Algebraic Theory of Semigroups I*, American Mathematical Society, Providence, R. I.
- [3] GUSTAFSON, W.H. (1973) What is the probability that two group elements commute. *Amer. Math. Monthly* **80** 1031–1034.
- [4] HOWIE, J.M. (1976) *An Introduction to Semigroup Theory*, Academic Press, London.
- [5] MACHALE, D. (1976) Commutativity in finite rings. *American Mathematical Monthly* **83** 30–32
- [6] NAGY, A. (2001) *Special Classes of Semigroups*, Kluwer Academic Publishers, Dordrecht, Boston, London.
- [7] NAGY, A. (2013) Left reductive congruences on semigroups. *Semigroup Forum* **87** 129–148.
- [8] NAGY, A. (2013) On faithful representations of finite semigroups S of degree $|S|$ over the fields. *International Journal of Algebra* **7** 115–129.

- [9] NAGY, A. and RÓNYAI, L. (2014) Finite Semigroups whose Semigroup Algebra over a Field Has a Trivial Right Annihilator. *International Journal of Contemporary Mathematical Science* **9** 25–36.
- [10] NAGY, A. (2015) Remarks on the paper "M. Kolibiar, On a construction of semigroups". *Periodica Mathematica Hungarica* **71** 261–264.
- [11] NAGY, A. (2015) Left Equalizer Simple Semigroups. *Acta Mathematica Hungarica* Online first, DOI: 10.1007/s10474-015-0578-6
- [12] PETRICH, M. (1977) *Lectures in Semigroups*, Akademie-Verlag-Berlin.